UNIVERSITY DEPARTMENT OF MATHEMATICS
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PAPER – VI | ASSIGNMENT – II | Complex Analysis

1. Problems on Cauchy Integral Formula
   (a) Evaluate the following integral along the positively oriented circle $\gamma : |z| = 2$

   i. $\oint_\gamma \frac{z^3+5}{z-i}dz$ ii. $\oint_\gamma \frac{1}{z^2+z+1}dz$ iii. $\oint_\gamma \frac{\sin z}{z^2+1}dz$

   (b) Evaluate the following integral in the given contours

   i. $\oint_{|z-4|=5} \frac{\cos z}{z}dz$ ii. $\oint_{|z-i|=1} \frac{z^2}{z^2+1}dz$

   iii. $\oint_{|z|=2} \frac{e^{i\pi z/2}}{z^2-1}dz$ iv. $\oint_{|z|=1} e^z z^{-3}dz$

   v. $\oint_{|z-1|=\frac{5}{2}} \frac{1}{(z-4)(z+1)^4}dz$ vi. $\oint_{|z|=2} \frac{\sin z}{(z^2-1)^2}dz$

2. Problems on Cauchy's Inequality
   (a) Let f be an entire function such that $|f(z)| \leq M|z|, \forall z \in \mathbb{C}$, where M is a fixed positive constant. Show that $f(z) = \alpha z$, where α is a complex constant.
   (b) Let f be an entire function and M be a constant such that

   $$|f(z)| \leq M|z|^{\frac{5}{4}}$$

   for all $z \in \mathbb{C}$. Show that we can find a constant $\alpha \in \mathbb{C}$ such that $f(z) = \alpha z^{\frac{5}{4}}$ for all $z \in \mathbb{C}$.

3. Using the maximum modulus theorem in complex analysis, find the maximum of $|f(z)|$ on $|z| \leq 1$, when $f(z) = z^2 - 3z + 2$.

4. Suppose f is analytic on the open disc $|z| < 1$ and satisfies $|f(z)| < M$ if $|z| < 1$. Suppose that $f(a) = 0$ for some $a, |a| < 1$. Then, show that

   $$|f(z)| \leq M \cdot \left|\frac{z-a}{1-\bar{a}z}\right|,$$

   where $\bar{a}$ is the complex conjugate of a.

   Hint: Define

   $$g(z) = \frac{f}{M} \circ \phi_a^{-1}(z), \quad \text{where } \phi_a(z) = \frac{z-a}{1-\bar{a}z},$$

   then $g : D \to D$, such that, $g(0) = 0$. Now apply the Schwartz Lemma.